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Geometry of Tangents, Local Polar Varieties and Chern Classes

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In this short report we give a summary without proofs of recent results concerning the geometry of analytic singularities. Most of these results were obtained in collaboration with B. Teissier.

0. Notations.

(0.1) Let $(X,0)$ be a germ of reduced equidimensional complex analytic space of complex dimension d. We may suppose that $(X,0)$ is embedded into $(\mathbb{C}^N,0)$. Let G be the Grassmann space of d-vector spaces in $\mathbb{C}^N$. Let X be a representant of $(X,0)$ in $\mathbb{C}^N$. We call Σ its singular locus and $X^\circ = X - \Sigma$ the subspace of non singular points of X. We have an analytic morphism $\gamma^\circ : X^\circ \to G$ defined by $\gamma^\circ(x) = T_xX$, where T_xX is the tangent space of X at a non singular point $x \in X$.

We consider $\tilde{X}$ the closure of the graph $Gr\gamma^\circ$ of γ° in $X \times G$. One knows that $\tilde{X}$ is a reduced analytic space (cf [11] lemma 3.9). The projection onto X defines $\nu: \tilde{X} \to X$ and the projection onto G defines $\gamma: \tilde{X} \to G$. We call ν <u>the Nash modification of X</u> and γ the <u>Gauss morphism of X</u>.

The set $|\nu^{-1}(0)|$ may be considered as the set of <u>limits</u>

of tangent spaces of X at 0 (cf [3],[4] for example). Actually $|\nu^{-1}(0)|$ defines a projective subvariety of G.

(0.2) In the same way we consider the restriction λ° to $X - \{0\}$ of the canonical map $\mathbb{C}^N - \{0\} \to \mathbb{P}^{N-1}$. Let X' be the closure of the graph of λ° in $X \times \mathbb{P}^{N-1}$. One knows that the projection onto X defines $e: X' \to X$, the blowing-up of the point $\{0\}$ in X, and the projection onto $\mathbb{P}^{N-1}$ defines the canonical map $\lambda: X' \to \mathbb{P}^{N-1}$. As above we may consider the set $|e^{-1}(0)|$ as the set of limits of secants of X at 0. Actually $e^{-1}(0)$ is the projective variety associated to the tangent cone $C_{X,0}$ of X at 0.

In the following we shall consider the blowing-up $\tilde{e}: \mathfrak{X} \to \tilde{X}$ of the analytic subspace $\nu^{-1}(0)$ of $\tilde{X}$. Thus we have a unique analytic morphism $\nu': \mathfrak{X} \to X'$ such that the following diagramm is commutative:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\tilde{e}"] \arrow[d, "\nu'"] & \tilde{X} \arrow[d, "\nu"] \\ X' \arrow[r, "e"'] & X \end{tikzcd}$$

Such a diagramm was considered in [2], [6]. Notice that actually $\mathfrak{X}$ is closed in $X \times G \times \mathbb{P}^{N-1}$ and, ν' and $\tilde{e}$ are respectively induced by the projections onto $X \times \mathbb{P}^{N-1}$ and $X \times G$.

(0.3) Remark: In this report we shall focus our attention on the case of complex hypersurfaces. In this case $N = d + 1$

and $G = \check{\mathbb{P}}^{d-1}$, the dual projective space of hyperplanes in $\mathbb{C}^d$.

In the case of complete intersections, if X is a sufficiently small representant of $(X,0)$, the Nash modification is the blowing-up of the Jacobian ideal of X. Thus, if X is the closed hypersurface defined by $f = 0$ in an open neighbourhood U of 0 in $\mathbb{C}^{d+1}$, then the Nash modification is the blowing-up of the ideal $J(f)$ generated by the partial derivatives $\partial f/\partial z_o, \ldots, \partial f/\partial z_d$ of f in $\mathcal{O}_X$. Moreover $\nu \circ \tilde{e} = e \circ \nu' = \eta$ is the blowing-up of the product of $J(f)$ and the maximal ideal $\mathfrak{m}$ defining $\{0\}$ in X.

1. <u>General results</u>.

In this paragraph we only suppose $(X,0)$ to be a reduced equidimensional complex analytic space of dimension d. For the proofs of the quoted results we mainly refer to [3], [4] or to [6].

(1.1) Let X be a sufficiently small representant of $(X,0)$ in $(\mathbb{C}^N,0)$. Then one may find a reduced complete intersection X_1 of dimension d in $\mathbb{C}^N$ such that $X \subset X_1$ (cf [6] 1.1.2). Let J_1 be the Jacobian ideal of X_1 and $\nu_1 : \tilde{X}_1 \to X_1$ the blowing-up of J_1 in X_1, i.e. the Nash modification of X_1. One can prove (cf [8]):

(1.1.1) <u>Theorem</u>: Let $\tilde{X}$ be the strict transform of X by ν_1, i.e. the closure of $\nu_1^{-1}(X - \Sigma_1)$ in $\tilde{X}_1$, then ν_1 induces an analytic morphism from $\tilde{X}$ onto X isomorphic over X with the Nash modification of X.

Thus the Nash modification of X may be defined by the blowing-up of the ideal $J_1\mathcal{O}_X$ of $\mathcal{O}_X$ generated by J_1.

(1.1.2) <u>Example</u>. Let P_1 and P_2 two planes of $\mathbb{C}^4$ in general position and $X = P_1 \cup P_2$. Thus 0 is an isolated singular point of X. It is easy to see that the Nash modification $\tilde{X}$ of X is non singular and has two connected components isomorphic by ν respectively to P_1 and P_2. Thus ν cannot be the blowing-up of an ideal with support in $\{0\}$.

We may choose X_1 to be $P_1 \cup P_2 \cup P_3 \cup P_4$ such that P_1, P_2, P_3, P_4 are the coordinate planes of a linear coordinate system of $\mathbb{C}^4$ at 0. Thus ν is the blowing-up of the subspace of the lines $D_1 \cup D_2 \cup D_3 \cup D_4$ of X where:

$$D_1 = P_1 \cap P_3$$
$$D_2 = P_1 \cap P_4$$
$$D_3 = P_2 \cap P_3$$
$$D_4 = P_2 \cap P_4$$

(1.2) In [12] H. Whitney has proved a lemma we may state in the following way:

(1.2.1) <u>Theorem</u> Let $\mathcal{H}$ be the subvariety of $G \times \mathbb{P}^{N-1}$ of couples (T,ℓ) such that $\ell \subset T$, then $\eta^{-1}(0) \subset \{0\} \times \mathcal{H}$

As a corollary we have:

(1.2.2) <u>Corollary</u> If $d = 1$, the set of limits of tangents coincides with the set of lines in the tangent cone.

We remind that this theorem of Whitney just says that the non singular part of X always has the Whitney condition along $\{0\}$ at 0. Recall that if Y is a non singular analytic space contained in X and X_o is an open analytic subset of the set of non singular points of X such that $\overline{X}_o = X$, we say that <u>X_o has the Whitney condition along Y at a point $0 \in Y$</u> if, for any sequence $\{x_n\}$ of X_o and any sequence $\{y_n\}$ of Y, such that:

1) $\lim x_n = 0$ and $\lim y_n = 0$
2) the limit of the lines $x_n y_n$ exists and: $\lim x_n y_n = \ell$
3) the limit of the tangent spaces $T_{x_n} X$ exists and: $\lim T_{x_n} X = T$

then $T \supset \ell$.

We say that <u>X_o has the Whitney condition along Y</u> if it has the Whitney condition along Y at any point of Y.

This notion was introduced by H. Whitney in [11].

(1.3) In [3] we show that:

(1.3.1) <u>Theorem</u> The limits of tangent spaces of the reduced tangent cone $|C_{X,0}|$ of X at 0 are limits of tangents of X at 0.

2. <u>Surfaces in $\mathbb{C}^3$</u>

(2.1) Before stating our main result in the general case of any hypersurface, we first show what is known in the case of the germ of a reduced surface $(X,0)$ in $(\mathbb{C}^3,0)$. In this case the linear planes of $\mathbb{C}^3$ define a 2-dimensional projective space we denote by $\check{\mathbb{P}}^2$. In [4] we prove:

(2.1.1) Theorem The variety of limits of tangents in $\check{\mathbb{P}}^2$ is the union of the set of planes tangent to the reduced cone and of a finite number of pencils of planes through lines in the tangent cone called exceptional tangents of X at 0.

In the case (X,0) is an isolated singularity, such a result was obtained in [3]. In [3] we moreover gave a precise geometric description of the exceptional tangents in relation with the blowing-up of X at 0.

From the above theorem one obtains:

(2.1.2) Corollary The set of limits of tangents of the germ of a reduced surface in $(\mathbb{C}^3,0)$ is finite if and only if the reduced tangent cone is a finite union of planes and (X,0) has no exceptional tangents.

In [4] we give a numerical criterion such that a germ of reduced surface in $(\mathbb{C}^3,0)$ has a finite number of limits of tangents.

(2.1.3) Examples:

1) The "swallow tail", i.e. the discriminant of the general polynomial of degree 4 for which the sum of roots is zero, has only one limit of tangents at 0.
2) If the singular locus of (X,0) is non singular at 0 and X is equisingular along it at 0, then the set of limit of tangents is finite.

   Actually the numerical criterion quoted above and Zariski's discriminant theorem about equisingularity (cf [14]) gives:

If the singular locus of the germ of a reduced surface $(X,0)$ in $(\mathbb{C}^3,0)$ is non singular, the limits of tangents of $(X,0)$ are finite <u>if and only if X is equisingular along its singular locus at 0</u>.

(2.2) In [5] we have studied the case of a germ of reduced surface $(X,0)$ in $(\mathbb{C}^3,0)$ with no exceptional tangent at 0. We obtained the following result:

(2.2.1) <u>Theorem</u> Let $(X,0)$ be the germ of a reduced surface $(X,0)$ in $(\mathbb{C}^3,0)$. If $(X,0)$ has no exceptional tangent at 0 and if the tangent cone $C_{X,0}$ of X at 0 is reduced, then $(X,0)$ has an equisingular deformation on its tangent cone.

Moreover if $(X,0)$ is an isolated singularity and if $(X,0)$ has no exceptional tangent at 0, then its tangent cone is reduced and the preceding result holds.

(2.2.2) <u>Examples</u>:

1) The "swallow tail" has no exceptional tangent but its tangent cone is not reduced.
2) The surface of $\mathbb{C}^3$ defined by

$$(x^2 + y^2 + z^2)^2 + z^5 = 0$$

(cf [5]) has no exceptional tangent but its tangent cone is given by $(x^2 + y^2 + z^2)^2 = 0$ and is not reduced.

(2.3) From [4] we can characterize geometrically the exceptional tangents of $(X,0)$. Let us consider projections p of $(X,0)$ onto $(\mathbb{C}^2,0)$. Let us denote by X^o the non singular part of X. Notice that if p is sufficiently general the critical space $C(p)$ of the restriction of p to X^o is either $\emptyset$ or non singular of dimension 1. We shall call Γ_p the closure of $C(p)$ in X. Thus Γ_p is either $\emptyset$ or a reduced curve of dimension 1. From [4] we obtain :

(2.3.1) Theorem: If the projection $p:(X,0) \to (\mathbb{C}^2,0)$ is sufficiently general, the tangent cone of Γ_p at 0 is the union of the exceptional tangents of $(X,0)$ and of the lines in the apparent contour of the projection of the reduced tangent cone $|C_{X,0}|$ onto $\mathbb{C}^2$ by p.

Thus we have (using (2.1.2)).

(2.3.2) Corollary If the projection $p:(X,0) \to (\mathbb{C}^2,0)$ is sufficiently general, the limits of tangents of $(X,0)$ are finite if and only if $\Gamma_p = \emptyset$.

If p is sufficiently general, we call $(\Gamma_p,0)$ a polar curve of $(X,0)$.

From theorem (2.3.1) we see that the exceptional tangents of $(X,0)$ are the lines of the tangent cone of a polar curve of $(X,0)$ which do not depend on the general projection $p:(X,0) \to (\mathbb{C}^2,0)$.

3. Complex hypersurfaces

(3.1) In this paragraph we shall state recent results of

B. Teissier and myself. In the whole paragraph $(X,0)$ denotes <u>the germ of a reduced hypersurface</u> in $(\mathbb{C}^{d+1},0)$. We shall use the notations of §0. We consider the diagramm of (0.2):

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\tilde{e}} & \tilde{X} \\ \downarrow \nu' & & \downarrow \nu \\ X' & \xrightarrow[e]{} & X \end{array}$$

where X is a sufficiently small representant of $(X,0)$.

(3.2) Let $\eta = \nu\circ\tilde{e} = e\circ\nu'$. We denote by $\mathcal{Y} = \eta^{-1}(0)$. Actually $\mathcal{Y}$ is the exceptional divisor of the blowing-up of the product of the maximal ideal $\mathfrak{m}$ defining $\{0\}$ and the jacobian ideal $J(f)$ of $(X,0)$, where $f = 0$ is an equation of $(X,0)$in $(\mathbb{C}^{d+1},0)$.

Let $(\mathcal{Y}_\alpha)_{\alpha\in A}$ be the irreducible components of $\mathcal{Y}$:

$$\mathcal{Y} = \bigcup_{\alpha\in A} \mathcal{Y}_\alpha$$

Let V_α be the image of $\mathcal{Y}_\alpha$ by ν'. It is a subvariety of the reduced projective variety $|Y'|$, where $Y' = e^{-1}(0)$(i.e. $\cong$ Proj $C_{X,0}$).

Notice that there are $\alpha_1,..,\alpha_k \in A$, such that $\dim V_{\alpha_i} = \dim |Y'|$ $(i=1,..,k)$ and $V_{\alpha_1} \cup .. \cup V_{\alpha_k} = |Y'|$. One may prove:

(3.2.1) <u>Theorem</u> For each component Y_i of $|Y'|$, there is only one $\alpha_i \in A$ such that $V_{\alpha_i} = Y_i$ and the variety of limits of tangents $(\cong|\tilde{Y}| = |\nu^{-1}(0)|)$ of $(X,0)$ is the union of the dual varieties of the V_α, i.e. the union of the subvarieties of $\check{\mathbb{P}}^d$ of hyperplanes which contain a tangent space to $V_\alpha (\alpha \in A)$.

(3.2.2) In the case of hypersurfaces we have a result similar to (2.3.1). To state it we need to introduce the notion of polar varieties in higher dimension. Let $p_k : (X,0) \to (\mathbb{C}^{k+1},0)$ be a projection onto $\mathbb{C}^{k+1} (1 \leq k \leq d)$. We denote by X^o the non singular part of a sufficiently small representant X of $(X,0)$. Let $C(p_k)$ be the critical space of the restriction of p_k to X^o. Then, if p_k is sufficiently general, $C(p_k)$ is either $\emptyset$ or Γ_k reduced with complex dimension k. We shall call Γ_k the closure of $C(p_k)$ in X. Obviously we can define such a Γ_k for any reduced analytic germ $(X,0)$ of pure dimension d when p_k is sufficiently general. We call Γ_k a <u>local k-polar variety of</u> $(X,0)$ when p_k is sufficiently general. We have the following theorem:

(3.2.3) <u>Theorem</u> If p_k is a sufficiently general projection, the reduced tangent cone of Γ_k at 0 is the union of the cones over the V_α $(\alpha \in A)$ such that $\dim V_\alpha + 1 = k$ and of the k-polar variety for p_k of the cones over the $V_\alpha (\alpha \in A)$ such that $\dim V_\alpha + 1 > k$.

(3.2.4) We notice that the cones over the $V_\alpha (\alpha \in A)$ such that $\dim V_\alpha + 1 = k$ are the components of the reduced tangent cone of a k-polar variety which do not depend on the general projection p_k.

From theorem (3.1.2) we can obtain a corollary similar to (2.3.2):

(3.2.5) <u>Corollary</u> The limits of tangents of $(X,0)$ are finite if and only if for any $1 \leq k \leq d-1$ and any general projection p_k, the polar variety Γ_k relative to p_k is $\emptyset$.

We have a formulation similar to the one of (2.1.2):

(3.2.6) Corollary The limits of tangents of $(X,0)$ are finite if and only if the reduced tangent cone of $(X,0)$ is the union of k hyperplanes and $\mathcal{Y}$ has k components.

We expect to have results similar to the ones of (2.2.1).

Besides of it we have the following equisingularity criterion:

(3.2.7) Proposition Suppose the singular locus of $(X,0)$ is non singular at 0 and has the codimension one in $(X,0)$. Then $(X,0)$ is equisingular along its singular locus at 0 if and only if the limits of tangents of X at 0 are finite.

4. General situation

(4.1) In this paragraph we shall give the results known in the general case of germ of reduced analytic space $(X,0)$ of pure dimension d.

First we have a result concerning the relation between the tangent cone of $(X,0)$ and the set of limits of tangent spaces of X at 0 (cf [3]) we have already quoted in (1.3.1).

Actually we even get a more precise result:

(4.1.2) Theorem There is a non void Zariski dense set U in the projective variety of lines of $|C_{X,0}|$ passing through 0 such that for any $\ell \in U$ and any sequence $x_n \neq 0$ of non singular points of X which tends to 0: $\lim_{n\to\infty} x_n = 0$ - and for which $\lim_{n\to\infty} 0x_n = \ell$ and $\lim T_{x_n} X = T$, then T is a limit of tangents of $|C_{X,0}|$ along ℓ.

(4.2) Actually, using the results of §3, it is better to make use of the following result (cf [6] 6.3.2).

(4.2.1) Theorem Let $(X,0)$ a germ of reduced analytic space of pure dimension d. Suppose that $(X,0)$ is contained in $(\mathbb{C}^N,0)$. For a sufficiently general projection $p: \mathbb{C}^N \to \mathbb{C}^{d+1}$, the images of local polar varieties of X at 0 are the local polar varieties of $p(X) = X_1$ at 0.

Using this theorem it is then easy to get a result similar to the one of (3.1.1), but we obtain the set of hyperplanes of $\mathbb{C}^N$ which contain a limit of tangent spaces of X at 0 and not the set of limits of tangent spaces of X at 0 itself.

(4.3) Now we have a relation between Chern classes for singular varieties defined and obtained by R. MacPherson in [7] and our geometric constructions (cf[6]).

Let $(X,0)$ be a reduced germ of analytic space of pure dimension d. For sufficiently general projections p_k $(1 \leq k \leq d)$ as defined in (3.1.2) the multiplicities at 0 of the corresponding polar varieties are analytic invariants of $(X,0)$. Let us denote: $m_k = m_0(\Gamma_k)$ - and $\mathcal{C}(X,0) = (m_1,\ldots,m_d)$

$$\mathrm{Eu}(X,0) = \sum_{k=1}^{d} (-1)^{d-k} m_k$$

We shall call $\mathrm{Eu}(X,0)$ the Euler obstruction of $(X,0)$(cf[7] and [6]).

Now we denote by $V_{oi} (i \in J_0)$ the irreducible components of X. Let $V_1 = \Sigma(X)$ be the singular locus of X and denote $V_{1i} (i \in J_1)$ the irreducible components of $\Sigma(X)$. For $x \in V_{1i}$ the value of $\mathcal{C}(X,x)$ has a constant value if and only if $x \in V_{1i} - V'_{1i}$, where

V'_{1i} is a <u>strict</u> analytic subset of V_{1i}. Let $V_2 = \Sigma(V_1) \bigcup_{i \in J_1} V'_{1i}$. Again we denote $V_{2i} (i \in J_2)$ the irreducible components of V_2. For $x \in V_{2i}$ the values of $\mathcal{C}(X,x)$ and $\mathcal{C}(V_{1j},x) (j \in J_1)$ are constant if and only if $x \in V_{2i} - V'_{2i}$, where V'_{2i} is a strict analytic subset of V_{2i}. Let $V_3 = \Sigma(V_2) \bigcup_{i \in J_2} V'_{2i}$. By induction we define $V_k (k \leq \ell)$ and $V_{ki} (k \leq \ell,\ i \in J_k)$. For $x \in V_i$ the values of $\mathcal{C}(X,x)$ and $\mathcal{C}(X_{rs},x)$ $(r \leq \ell - 1,\ s \in J_r)$ are constant if and only if $x \in V_{\ell i} - V'_{\ell i}$, where $V'_{\ell i}$ is a strict analytic subset of $V_{\ell i}$. Then $V_{\ell+1} = \Sigma(V_\ell) \bigcup_{i \in J_\ell} V'_{\ell i}$ and the $V_{\ell+1 i} (i \in J_{\ell+1})$ are the irreducible components of $V_{\ell+1}$. Obviously this process ends after a finite number of steps, as $V_{\ell+1} = \emptyset$ for some ℓ.

Now let n_{ij} be a family of integers defined inductively by:

$$n_{00} = 1$$

$$k \geq 1,\ \ell \in J_k:\ \sum_{\substack{0 \leq i \leq k-1 \\ \ell \in J_i}} n_{ij} \operatorname{Eu}(V_{ij},x) + n_{k\ell} = 1 \quad \text{with} \quad x \in V_{k\ell} - V'_{k\ell}$$

Now the cycle defined by R. MacPherson which gives the local chern class of $(X,0)$ is $\sum_{\substack{i \\ j \in J_i}} n_{ij} V_{ij}$.

(4.4) A result stated by B. Teissier in [10] gives:

(4.4.1) <u>Theorem</u> The stratification of X defined by the strata $F_{ij} = V_{ij} - \bigcup_{k>i} V_{k\ell}$ satisfies the Whitney condition. Moreover any stratification of X in which the singular locus, the singula locus of the singular locus etc.., are union of strata and which satisfies the Whitney condition is finer than this stratification.

(4.4.2) This result shows that the Whitney condition is "analytic". In the case of projective varieties the F_{ij} obtained in the above construction are quasi projective.

For example this helps to have a constructive way to define the generic hyperplanes of [13] to obtain a computation of the fundamental group of the complement of a projective hypersurface (cf[1]).

(4.5) In the case one considers cones over projective varieties, the local polar varieties we define are cones and give a local version of the results obtained and quoted by R. Piene in [9].

(4.6) In [6] (Appendice) we obtain a theorem which gives a relation between the limits of tangent spaces on any equidimensional reduced analytic space and the ones of its generic projection as a hypersurface. Namely we have:

(4.6.1) <u>Theorem</u>: Let $(X,0)$ be a germ of equidimensional reduced analytic space of dimension d embedded in $(\mathbb{C}^N,0)$. There is an open dense Zariski subset $\mathcal{U}$ of the Grassmann space of linear projections $p_o : \mathbb{C}^N \to \mathbb{C}^{d+1}$ such that, if $p_o \in \mathcal{U}$:

a) The induced analytic morphism by p_o from X onto $X_1 = p_o(X)$ is finite and bimeromorphic;

b) In the Grassmann variety G of d-vector spaces on $\mathbb{C}^N$, the (Schubert) variety C of d-vector spaces not transverse to $\operatorname{Ker} p_o$ meets any component Y_k of the space $|Y|$ of limits of tangent spaces of $(X,0)$ in an analytic subset of codimension 2 on Y_k or void;

c) The morphism $p : X \to X_1$, induced by p_o, is finite and defines an analytic morphism $\tilde{p}$ from $\tilde{X}-\gamma^{-1}(C)$ into $\tilde{X}_1$ which is finite (with $\gamma:\tilde{X} \to G$ being the Gauss morphism of $(X,0)$ and $\tilde{X}_1$ the Nash modification space of X_1); moreover $\tilde{p}$ induced an analytic isomorphism of the normalization of $\tilde{X} - \gamma^{-1}(C)$ onto the normalization of $\tilde{p}(\tilde{X} - \gamma^{-1}(C))$.

This theorem should allow to generalize the results of §3.

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